State 5 is completely defined.

Step 3: The reflected shock (b-c) intersects the interface at point c. Initial conditions:

$$[L]: p_2, \rho_2, u_2; \quad [R]: p_5, \rho_5, u_5 = 0$$

Solution defines states 4 and 6.

Step 4: Reflected shock (c-e) is re-reflected from the wall at point e. Initial conditions:

$$[L]: p_6, \rho_6, u_6; \quad [R]: p_6, \rho_6, -u_6$$

Solution defines state 8.

Step 5: Shock (e-f) hits interface at point f. Initial conditions:

$$[L]: p_6, \rho_6, u_6; \quad [R]: p_6, \rho_6, -u_6$$

Solution defines state 7.

Step 6: Air in the rotor cells is released to exhaust (ambient) conditions. Initial conditions:

$$[L]: p_{amb}, \rho_{amb}, u_{amb} = 0; \quad [R]: p_7, \rho_7, u_7 = 0$$

Solution defines state 9.

Step 7: Rarefaction wave (g-h) intersects interface at point h. Initial conditions:

$$[L]: p_9, \rho_9, u_9; \quad [R]: p_8, \rho_8, u_8 = 0$$

Solution defines states 9 and 10.

Step 8: Rarefaction wave hits the wall at point *i*, and is reflected in like sense. Initial conditions:

$$[L]: p_{10}, \rho_{10}, -u_{10}; \quad [R]: p_{10}, \rho_{10}, u_{10}$$

Solution defines state 1.

This should match with the original state 1 for "cycle closure," which is required if the rotor is to operate continuously.

Table 1 gives the values computed for the cycle and with the procedure described above. The pressures are static values and the velocities are referred to the rotor.

Clearly, a mismatch of rotor speed and/or inlet conditions from design point values will cause new waves to be generated which have to be carried through the entire cycle in order to assess their overall effect on the performance. The Riemann program is particularly useful for preliminary design because of its "building block" approach, which allows walking through any wave configuration state by state. Once a viable cycle is established, a detailed flow solver can be applied to incorporate effects of friction, heat transfer and the finite times taken for cell opening and closing.

Conclusions

The calculation procedure in the example above required only minutes to carry out, with minimal requirements for computer time or storage. (On average, a typical "Riemann Step" calculation required 0.02 s of CPU time on an IBM 370-3033AP computer.) The Riemann problem solver code, therefore, gives a fast, efficient, and unified approach to carry out the preliminary design of wave rotor devices with diverse wave structures and pressure ratios.

It is noted that the code may be coded easily on any home computer and does not require external hardware or software libraries.

Acknowledgment

This work was supported in part by ONR Contract N0001482SR20232.

References

[1] Pearson, R. D., "Pressure Exchangers and Pressure Exchange Engines," Ch. 16, *Thermodynamics and Gas Dynamics of I. C. Engines,* compiled by D. E. Winterbone and S. C. Low, (to be published).

[2] Rose, P. H., "Potential Applications of Wave Machinery to Energy and Chemical Processes," presented at the 12th International Symposium on Shock Tubes and Waves, Jerusalem, Israel, July 16-19, 1979.

[3] Croes, Ni., "The Principle of the Pressure-Wave Machine as Used for Charging Diesel Engines," Dept. Comprex Research, A. G. Brown Boveri and Cie, Baden, Switzerland. Available from authors on request.

[4] Pearson, R. D., personal communication, Alpe Thermo Products Bristol, England, Aug. 1982.

[5] Courant, R. and Friedrichs, K. O., *Supersonic Flow and Shock Waves,* Interscience Publishers Inc., New York, 1948.

[6] Teng, Z-H., Liu, T-P., and Chorin, A. J., "Riemann Problems for Reacting Gas, with Applications to Transition," Center for Pure and Applied Mathematics, University of California, Berkeley, Calif., Rept. PAM-29, Jan. 1981.

Classical Normal Modes in Asymmetric Nonconservative Dynamic Systems

Mehdi Ahmadian* and Daniel J. Inman†
State University of New York at Buffalo
Buffalo, New York

Introduction

THE dynamic behavior of a general linear discrete system can be described by the vector differential equation

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = 0 \tag{1}$$

where M, C, and K are mass, damping, and stiffness matrices, respectively. The usual treatment of these systems assumes that Eq. (1) is symmetric. Although this assumption is justified for passive systems, in many problems of interest in aeronautics, ship vibrations, and active control of large space structures, Eq. (1) cannot be presented in a symmetric form. This motivates the study of the behavior and properties of this class of problems and also an attempt to derive relations similar to those of symmetric systems.

It is well known that a passive conservative system (i.e., $C=0$, $M=M^T$, and $K=K^T$) possesses normal modes. Rayleigh[1] has shown that a passive nonconservative system possesses normal modes if the damping matrix is proportional to the mass and stiffness matrices, i.e., $C=\alpha M+\beta K$. Furthermore, it has been shown by Caughey and O'Kelly[2-4] that a damped linear symmetric system possesses normal modes if

Received May 5, 1983; revision received June 28, 1983.

*Graduate Research Assistant, Department of Mechanical and Aerospace Engineering. Student Member AIAA.

†Assistant Professor, Department of Mechanical and Aerospace Engineering. Member AIAA.

and only if $CM^{-1}K=KM^{-1}C$. Without mentioning asymmetric systems, classical symmetric systems are studied exclusively in Ref. 2. Although in Ref. 3 O'Kelly talks extensively about normal modes in passive systems, the existence of normal modes in general asymmetric systems is also briefly mentioned. The conditions presented by O'Kelly require C and K to be normal matrices. This restricts the applicability of this result to systems where the C and K matrices are unitarily similar to the diagonal matrices of their eigenvalues. According to Ref. 4, possession of classical normal modes is conditional to the existence of M^{-1} and a complete set of eigenvectors common to both the matrices $M^{-1}K$ and $M^{-1}C$. Checking these two assumptions requires the eigenproblem solution for the $M^{-1}K$ and $M^{-1}C$ matrices and, again, is not convenient.

The intent of the work presented here is twofold. First, it is intended to generalize the concept of Rayleigh damping for general asymmetric systems with the assumption that the matrix $M^{-1}K$ is simple and M is nonsingular. Second, conditions are developed for the existence of classical normal modes in a subclass of general systems (i.e., symmetrizable systems) where, unlike the results presented in Refs 2-4, neither the normality of C and K nor the eigenproblem solution of the $M^{-1}C$ and $M^{-1}K$ matrices is required.

Results

Assuming M to be nonsingular and premultiplying by M^{-1}, Eq. (1) can be written as

$$I\ddot{x}(t)+\tilde{C}\dot{x}(t)+\tilde{K}x(t)=0 \tag{2}$$

where $\tilde{C}=M^{-1}C$, $\tilde{K}=M^{-1}K$, I denotes the identity matrix and Eqs. (2) and (1) have the same eigenvalues.

Several of the results for normal modes of symmetric systems are still valid with minor adjustment for the asymmetries. First, it will be shown in the following theorem that the concept of Rayleigh damping holds true for asymmetric systems with simple $\tilde{K}$ matrix.

Theorem 1: If $\tilde{K}$ is simple and $\tilde{C}$ is proportional to the I and $\tilde{K}$ matrices (i.e., $\tilde{C}=\alpha I+\beta\tilde{K}$ or $C=\alpha M+\beta K$), then there exists a nonsingular linear transformation that decouples Eq. (2)

Proof: Substituting the assumed form $\tilde{C}$ into Eq. (2) results in

$$I\ddot{x}(t)+(\alpha I+\beta\tilde{K})\dot{x}(t)+\tilde{K}x(t)=0 \tag{3}$$

Since $\tilde{K}$ is a simple matrix, Eq. (3) can be transferred into a diagonal form by letting $y(t)=\phi x(t)$ and premultiplying by ϕ^{-1}

$$I\ddot{y}(t)+(\alpha I+\beta\phi^{-1}\tilde{K}\phi)\dot{y}(t)+\phi^{-1}\tilde{K}\phi y(t)=0 \tag{4}$$

where ϕ is the modal matrix of $\tilde{K}$ and the matrix $\phi^{-1}\tilde{K}\phi$ and Eq. (4) is in diagonal form. Thus, the asymmetric system described by Eq. (2) possesses normal modes if the asymmetric damping matrix is proportional to the mass and stiffness matrices (Rayleigh damping).

Caughey and O'Kelly have shown in Ref. 2 that a symmetric system possesses normal modes if and only if $CM^{-1}K=KM^{-1}C$. O'Kelly[3] talks extensively about normal modes in passive linear systems and their application in solution of forced response. However, the existence of normal modes in general asymmetric systems is only briefly mentioned in this reference. With no explicit proof, it is stated: "In many systems M is symmetric and positive definite and K and C are both normal matrices. In this case it is necessary and sufficient for $M^{-1}K$ and $M^{-1}C$ to commute for system diagonalizability in n-space...." According to Ref. 5, a matrix A is defined normal if and only if it commutes with its conjugate transpose ($AA^*=A^*A$). This reduces the use of these results to those systems where the $M^{-1}C$ and $M^{-1}K$ matrices are unitarily similar to the diagonal matrices of their eigenvalues.

In Ref. 4, normal modes for a general system and the application of normal modes in solving for the forced response of a linear asymmetric system is studied. It is shown that for the case that $M^{-1}K$ has distinct eigenvalues, the Caughey series, i.e.,

$$M^{-1}C=\sum_{i=0}^{n-1}\alpha_i(M^{-1}K)^i$$

is both necessary and sufficient for existence of classical normal modes. This condition is only sufficient if $M^{-1}K$ has repeated roots. As an alternative approach, the conditions for a system to possess classical normal modes are stated as: 1) M^{-1} exists and 2) $M^{-1}K$ and $M^{-1}C$ have the same complete set of eigenvectors, which involves the eigenproblem solution for the matrices $M^{-1}C$ and $M^{-1}K$.

Here, it is intended to show that the same results developed in Ref. 2 can be stated for a subclass of general systems. The class of asymmetric systems considered are those reducible to a symmetric form (i.e., symmetrizable systems). Unlike the results developed in Refs. 3 and 4, these results require neither the normality of C and K nor the eigenproblem solution of $M^{-1}C$ and $M^{-1}K$ matrices.

Before actually discussing the theorem, a definition of symmetrizable matrices and systems is in order. A matrix is said to be symmetrizable if and only if it can be expressed as the product of two symmetric matrices one of which is positive definite. In Ref. 6, Tausskey shows that a matrix is symmetrizable if and only if at least one of the following hold.

1) It is similar to a symmetric matrix.
2) It has real eigenvalues and a full set of eigenvectors.
3) It is similar to its transpose.
4) It becomes symmetric when multiplied by a suitable positive-definite matrix.

A symmetrizable asymmetric system is defined as one which is similar to a symmetric system. According to Ref. 7, an asymmetric system described by Eq. (2) is symmetrizable if and only if

$$\tilde{C}=S_1S_2,\qquad S_1>0,\quad S_2=S_2^T$$

and

$$\tilde{K}=T_1T_2,\qquad T_1>0,\quad T_2=T_2^T$$

have at least one factorization in common (e.g., $S_1=T_1$).

Theorem 2: A symmetrizable asymmetric system, described by Eq. (1), possesses normal modes if and only if $CM^{-1}K=KM^{-1}C$.

Proof: Since $\tilde{C}$ and $\tilde{K}$ are symmetrizable matrices, they can be written as

$$\tilde{C}=S_1S_2,\qquad S_1^T=S_1>0,\qquad S_2=S_2^T$$
$$\tilde{K}=T_1T_2,\qquad T_1^T=T_1>0,\qquad T_2=T_2^T \tag{5}$$

According to Ref. 7, since the above system is symmetrizable, there exits at least one factorization of $\tilde{C}$ and $\tilde{K}$ such that $S_1=T_1$. Considering this fact and substituting Eq. (5) into Eq. (2) results in

$$I\ddot{x}(t)+S_1S_2\dot{x}(t)+S_1T_2x(t)=0 \tag{6}$$

The system described by Eq. (6) is similar to a symmetric system which can be obtained by letting $y(t)=S_1^{1/2}x(t)$ and premultiplying Eq. (6) by $S_1^{-1/2}$, i.e.,

$$\ddot{y}(t)+S_1^{1/2}S_2S_1^{1/2}\dot{y}(t)+S_1^{1/2}T_2S_1^{1/2}y(t)=0 \tag{7}$$

According to Caughey and O'Kelly,[2] this symmetric equation possesses normal modes if and only if the coefficient matrices of $\dot{y}(t)$ and $y(t)$ commute. Implying

$$S_I^{1/2} S_2 S_I^{1/2} S_I^{1/2} T_2 S_I^{1/2} = S_I^{1/2} T_2 S_I^{1/2} S_I^{1/2} S_2 S_I^{1/2} \tag{8}$$

Pre- and postmultiplying the preceding matrix equation by $S_I^{1/2}$ and $S_I^{-1/2}$, respectively, results in

$$S_I S_2 S_I T_2 = S_I T_2 S_I S_2 \tag{9}$$

or

$$\tilde{C}\tilde{K} = \tilde{K}\tilde{C} \tag{10}$$

which implies

$$CM^{-1}K = KM^{-1}C$$

Hence, the asymmetric system described by Eq. (2) can be decoupled if and only if the preceding relation holds true (the extension of Caughey and O'Kelly's normal mode theorem).

The validity of the given results are illustrated by the following examples.

Example 1

Consider the discretized model of a damped flexible, rotatory shaft presented in Ref. 8. A point mass is attached to a coordinate system which rotates at a constant angular velocity Ω. To make the problem more comprehensive we assume the existance of a nonconservative force which is proportional to the radial distance of the mass from the origin and perpendicular to the radius vector. Such a force can be visualized as arising in a rotating fluid or in an electromagnetic field. The described model is shown more explicitly in a figure in Ref. 8.

The governing equation of motion for the shaft is

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{Bmatrix} + \begin{bmatrix} c_1 & -2m\Omega \\ 2m\Omega & c_2 \end{bmatrix} \begin{Bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{Bmatrix} + \begin{bmatrix} k_1 - m\Omega^2 & Fr \\ -Fr & k_2 - m\Omega^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 0$$

where U_1 and U_2 represent the displacements in the direction of the two rotating coordinate axes. Letting $m=1$, $k_1=1$, $k_2=4$, $c_1=1$, $c_2=4$, $Fr=-1$, $\Omega=0.5$, the preceding equation can be written as

$$\begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} \begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{Bmatrix} + \begin{bmatrix} 1.00 & -1.00 \\ 1.00 & 4.00 \end{bmatrix} \begin{Bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{Bmatrix} + \begin{bmatrix} 0.75 & -1.00 \\ +1.00 & 3.75 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 0 \tag{11}$$

The given damping matrix is proportional to the mass and stiffness matrices, the stiffness matrix is simple and reducible to a diagonal form via the similarity transformation

$$S = \begin{bmatrix} 0.3568 & 0.9342 \\ -0.9342 & -0.3568 \end{bmatrix}$$

According to the theorem 1 normal modes are existent and the system can be decoupled by letting $V = SU$ and premultiplying Eq. (11) by S^{-1}.

$$\begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} \begin{Bmatrix} \ddot{V}_1 \\ \ddot{V}_2 \end{Bmatrix} + \begin{bmatrix} 3.62 & 0.00 \\ 0.00 & 1.38 \end{bmatrix} \begin{Bmatrix} \dot{V}_1 \\ \dot{V}_2 \end{Bmatrix} + \begin{bmatrix} 3.36 & 0.00 \\ 0.00 & 1.13 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = 0$$

which is as predicted by theorem 1.

Example 2

Consider the following hypothetical asymmetric system.

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} 2.0 & -5.0 & -6.0 \\ 1.0 & 8.0 & 0.0 \\ -2.0 & -3.0 & -4.0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} -1.0 & 7.2 & 9.6 \\ 0.0 & -5.8 & -2.4 \\ 2.0 & 1.6 & -2.2 \end{bmatrix} x(t) = 0 \tag{12}$$

where the damping and stiffness matrices commute and the system is symmetrizable; i.e., $CK = KC$.

$$C = \begin{bmatrix} 2.0 & -5.0 & -6.0 \\ 1.0 & 8.0 & 0.0 \\ -2.0 & -3.0 & 4.0 \end{bmatrix} = S_I S_2 = \begin{bmatrix} 2.0 & -1.0 & 0.0 \\ -1.0 & 2.0 & -1.0 \\ 0.0 & -1.0 & 1.0 \end{bmatrix} \times \begin{bmatrix} 1.0 & 0.0 & -2.0 \\ 0.0 & 5.0 & 2.0 \\ -2.0 & 2.0 & 6.0 \end{bmatrix}, \quad S_I > 0, \quad S_2 = S_2^T$$

and

$$K = \begin{bmatrix} -1.0 & 7.2 & 9.6 \\ 0.0 & -5.8 & -2.4 \\ 2.0 & 1.6 & -2.2 \end{bmatrix} = S_I T_2 = \begin{bmatrix} 2.0 & -1.0 & 0.0 \\ -1.0 & 2.0 & -1.0 \\ 0.0 & -1.0 & 1.0 \end{bmatrix} \times \begin{bmatrix} 1.0 & 3.0 & 5.0 \\ 3.0 & -1.2 & 0.4 \\ 5.0 & 0.4 & -1.8 \end{bmatrix}, \quad S_I > 0, \quad T_2 = T_2^T$$

According to the theorem 2, the above system possesses normal modes, which can be shown by calculating the modal matrix of K

$$S = \begin{bmatrix} 0.103 & 0.873 & 0.916 \\ -0.811 & -0.436 & -0.116 \\ 0.575 & -0.218 & 0.383 \end{bmatrix}$$

letting $y = Sx$, and premultiplying Eq. (12) by S^{-1}. This results in

$$\begin{bmatrix} 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 1.00 \end{bmatrix} \ddot{y}(t) + \begin{bmatrix} 7.87 & 0.00 & 0.00 \\ 0.00 & 6.00 & 0.00 \\ 0.00 & 0.00 & 0.13 \end{bmatrix} \dot{y}(t) + \begin{bmatrix} 7.00 & 0.00 & 0.00 \\ 0.00 & -4.10 & 0.00 \\ 0.00 & 0.00 & 2.10 \end{bmatrix} y(t) = 0$$

Thus, the system with commuting damping and stiffness matrices is similar to a diagonal system and possesses normal modes as predicted by theorem 2.

Discussion

The concepts of classical normal modes for a general discrete linear system were defined. The well-known theorem developed by Rayleigh for existence of normal modes in symmetric damped systems was extended to a more general class of dynamic systems, i.e., asymmetric systems with simple coefficient matrices. Some results developed by Caughey and O'Kelly on classical normal modes in symmetric and asymmetric systems were discussed and used to generate results on classical normal modes, similar to those available for symmetric second-order systems.

It should be noted that the results developed here are not necessarily more general than those developed by Caughey and O'Kelly. However, the new conditions for classical normal modes in asymmetric systems, presented here, can be checked in a systematic way and, hence, may be more computationally attractive.

The discretized model of a damped, flexible, rotatory shaft, and a hypothetical three-degree-of-freedom asymmetric system were used to illustrate the developed results.

Acknowledgment

This work was performed under National Science Foundation Grant MEA-8112826.

References

[1] Lord Rayleigh, *The Theory of Sound*, Vol. 1, Dover Press, New York, 1945, p. 130.

[2] Caughey, T. K. and O'Kelly, M. F. J., "Classical Normal Modes in Damped Linear Dynamic Systems," *ASME Journal of Applied Mechanics*, Vol. 32, June 1965, pp. 583-588.

[3] O'Kelly, M. E. J., "Vibration of Viscously Damped Linear Dynamic Systems," Ph.D. Dissertation, California Institute of Technology, May 1964, pp. 47-50.

[4] Caughey, T. K. and O'Kelly, M. E. J., "General Theory of Vibration of Damped Linear Systems," Dynamics Laboratory Rept., California Institute of Technology, June 1963, pp. 14-41.

[5] Lancaster, P., *Eigenvalues and Eigenvectors, Theory of Matrices*, 1st ed., Academic Press, Inc., New York, 1969, pp. 81,82.

[6] Tausskey, O., "Positive Definite Matrices and Their Role in the Study of the Characteristic Roots of General Matrices," *Advances in Mathematics*, Vol. 2, 1968, pp. 175-186.

[7] Inman, D. J., "Dynamics of Asymmetric Non-Conservative Systems," *Journal of Applied Mechanics*, Vol. 150, March 1983, pp. 199-203.

[8] Shieh, R. C. and Masur, E. F., "Some General Principle of Dynamic Instability of Solid Bodies," *Zeitschrift fuer Angewandte Mathematik und Physik*, Vol. 19, Sept. 1968, pp. 927-941.

Thermal Effect on Axisymmetric Vibrations of an Orthotropic Circular Plate of Variable Thickness

J. S. Tomar* and A. K. Gupta†
University of Roorkee, Roorkee, India

Introduction

Much work has been done on the vibrations of orthotropic circular plates,[1-3] but none of it has considered the thermal effect on such vibrations. It is well known[4] that, in the presence of a constant thermal gradient, the elastic coefficients of homogeneous materials become functions of the space variables. Fanconneau and Marangoni[5] have investigated the effect of the nonhomogeneity caused by a thermal gradient on the natural frequencies of simply supported plates of uniform thickness. Recently, Tomar and Tewari[6] have studied the effect of the thermal gradient on the frequencies of a circular plate of linearly varying thickness.

Thermally induced vibrations of elastic plates are of great interest in aircraft and machine designs and also in chemical, nuclear, and astronautical engineering. Nonisotropic plates of nonuniform thickness are being widely used in the design of modern missiles, space vehicles, aircraft wings, and numerous composite engineering machines. The analysis presented here studies the effect of a constant thermal gradient on frequencies of an orthotropic circular plate of variable thickness. The deflection function is expressed here in the form of an infinite series from which frequencies corresponding to the first three vibration modes are obtained for various values of the thickness variation, taper constant, and temperature gradient.

Received March 8, 1983; revision received Aug. 2, 1983.

*Professor, Department of Mathematics.
†Research Fellow, Department of Mathematics.

Analysis and Equation of Motion

It is assumed that the circular plate of orthotropic materials is subjected to a steady temperature in the radial direction

$$T = T_0(1-R) \tag{1}$$

where T denotes the temperature excess above the reference temperature at any point at a distance $R=r/a$ from the center of the circular plate of radius a and T_0 denotes the temperature excess at $r=0$ above the reference temperature at any point on the circumference of the circular plate, i.e., at $r=a$ or $R=1$.

The temperature dependence of Young's moduli in the r and Θ directions for most of the orthotropic material is given by

$$E_r(T) = E_1(1-\gamma T), \qquad E_\Theta(T) = E_2(1-\gamma T) \tag{2}$$

where E_1 and E_2 are the values of Young's moduli, respectively along the r and Θ directions at the reference temperature, i.e., at $T=0$.

Taking the reference temperature as that at the edge of the circular plate, i.e, at $R=1$, Young's moduli in view of Eqs. (1) and (2) becomes

$$E_r(R) = E_1[1-\alpha(1-R)], \qquad E_\Theta(R) = E_2[1-\alpha(1-R)] \tag{3}$$

where $\alpha = \gamma T_0$ $(0 \leq \alpha \leq 1)$, which is a parameter known as the temperature gradient.

The governing differential equation of axisymmetric motion of an orthotropic circular plate of variable thickness is[2]

$$\begin{aligned} &D_r w_{,rrrr} + 2[(D_r + rD_{r,r})/r]\,w_{,rrr} \\ &+ [(-D_\Theta + r(2+\nu_\Theta)D_{r,r} + r^2 D_{r,rr})/r^2]\,w_{,rr} \\ &+ [(D_\Theta - rD_{\Theta,r} + r^2\nu_\Theta D_{r,rr})/r^3]\,w_{,r} + \rho h w_{,tt} = 0 \end{aligned} \tag{4}$$

where $D_r = E_r h^3/12(1-\nu_r\nu_\Theta)$ and $D_\Theta = E_\Theta h^3/12(1-\nu_r\nu_\Theta)$. Also, ν_r and ν_Θ are the Poisson's ratio in the r and Θ directions, respectively, w the transverse deflection, ρ the mass density per unit volume, t the time, and h the thickness. A comma followed by a subscript denotes partial differentiation with respect to that variable.

Since the axis of the plate coincides with the radial direction, one may find that thickness h, D_r, and D_Θ of the plate become functions of r alone. For free transverse vibrations of